Leftovers of the last class:

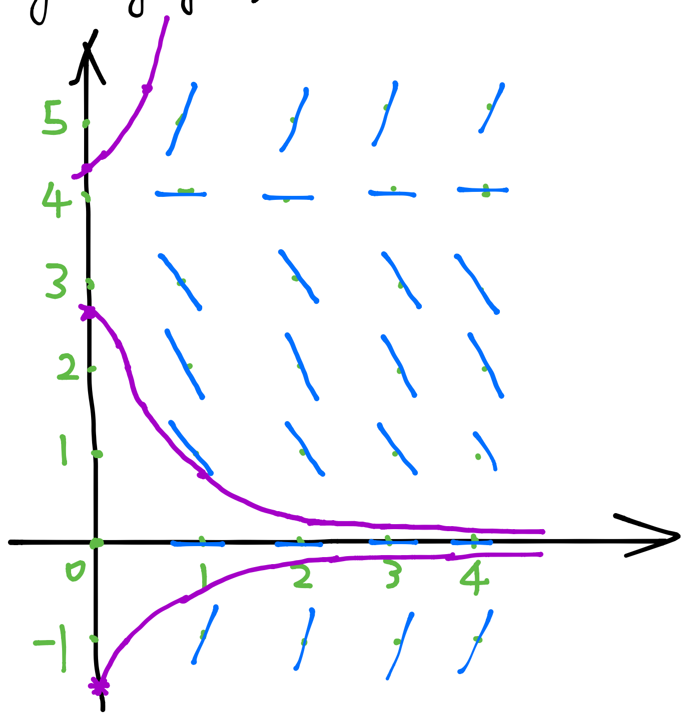
$*$ Remarks on direction fields

① It is possible to draw dir. field for arbitrary first order ODEs. Check MIT Lecture 1.

② $t \to \infty$ behavior

Example: $y' = y(y-4)$

y_0	$f(t_0, y_0)$
5	5
4	0
3	-3
2	-4
1	-3
0	0
-1	5



If start at $y_0 > 4$. $t_0 = 0$, i.e. if $y(0) > 4$, $y(t) \to +\infty$

If $y(0) \in (0, 4)$, $t \to \infty$, $y(t) \to 0$

If $y(0) < 0$, $t \to \infty$, $y(t) \to 0$

If $y(0) = 4$, $y(t) = 4$ no matter how t changes Equilibrium

If $y(0) = 4$, $y(t) = 0$ no matter how t changes Equilibrium

* How to check a given function is a sol'n to an ODE

Ans: Put this func. into the ODE, see if equality holds

Example: $\varphi(t) = e^{2t}$. $y'' - y' - 2y = 0$?

LHS $= \varphi'' - \varphi' - 2\varphi = 4e^{2t} - 2e^{2t} - 2e^{2t} = 0 =$ RHS ✓

Example $\varphi(t) = e^{t^2}$. $y' - 2ty = 0$

$\varphi'(t) = 2te^{t^2}$, $\varphi' - 2t\varphi = 2te^{t^2} - 2te^{t^2} = 0 =$ RHS ✓

Example: $\varphi(t) = e^{t^2}$. $y' - y = e^{t^2}$

$\varphi' = 2te^{t^2}$. $\varphi' - \varphi = (2t-1)e^{t^2} \neq e^{t^2} =$ RHS ✗

* Classification:

In this class, we will focus on linear ODEs.

$F(t, y', y'', \cdots, y^{(n)}) = 0$. $\frac{\partial F}{\partial y^{(i)}}$ is a function depends only on t

i.e. $\frac{\partial F}{\partial y^{(i)}} = f_i(t)$. independent of $y, y', \cdots, y^{(n)}$.

Or simpler, F is linear in $y, y', \cdots, y^{(n)}$. (no need to be linear in t)

Examples: $y'' - 2y' + y = \sqrt{t}$ $F = y'' - 2y' + y - \sqrt{t}$

$\frac{\partial F}{\partial y} = 1$, $\frac{\partial F}{\partial y'} = -2$, $\frac{\partial F}{\partial y''} = 1$. indep. y, y', y''. linear.

Example: $y' = \sin y$. $F = y' - \sin y$, $\frac{\partial F}{\partial y} = \cos y$ dep. on y.

$\Rightarrow$ nonlinear.

Example: $y'' - \sqrt{t}\, y' + \sin(t^2) y = t^3$ $F = y'' - \sqrt{t} y' + \sin(t^2) y - t^3 = 0$

$\frac{\partial F}{\partial y''} = 1$, $\frac{\partial F}{\partial y'} = -\sqrt{t}$ $\frac{\partial F}{\partial y} = \sin(t^2)$ indep. of y, y', y''

$\Rightarrow$ linear.

Assertion: Every n-th order linear ODE can be written as

$$y^{(n)} + a_1(t) y^{(n-1)} + a_2(t) y^{(n-2)} + \cdots + a_{n-1}(t) y' + a_n(t) y = g(t)$$

— Standard form!

(Coefficient of the highest derivitive $= 1$)

All other ODEs are called nonlinear.

We know a lot about linear ODE, very few about nonlinear ODEs.

First order linear ODE.

Standard form: $y' + p(t)y = g(t)$

Integrating factor: $\mu(t) = e^{\int p(t)\,dt}$

General Solution: $y = \dfrac{\int \mu(t) g(t)\,dt + C}{\mu(t)}$

How comes the formula:

· Observe: $\left(y' + p(t)y\right)e^{\int p(t)\,dt} = \left(e^{\int p(t)\,dt}\,y\right)'$

In general, any $\mu(t)$ making

$$\mu(t)\,y'(t) + \mu(t)\,p(t)\,y(t) = \left(\mu(t)\,y(t)\right)'$$

is called an integrating factor. To satisfy this eqn.

$\text{RHS} = \mu'(t)\,y(t) + \mu(t)\,y'(t)$

$\text{LHS} = \text{RHS} \Rightarrow \mu(t)\,p(t)\,y(t) = \mu'(t)\,y(t)$

$$\Rightarrow \mu(t)p(t) = \mu'(t) \Rightarrow \frac{\mu'(t)}{\mu(t)} = p(t)$$

Integrate: $\ln|\mu(t)| = \int p(t)\,dt$

$$\mu(t) = e^{\int p(t)\,dt}.$$

Now that $\mu(t)\,y'(t) + \mu(t)\,p(t)\,y(t) = (\mu(t)\,y(t))'$

they shall be equal.

Meanwhile $\mu(t)\,(y' + p(t)y) = \mu(t)\,g(t)$

$$\Rightarrow (\mu(t)\,y(t))' = \mu(t)\,g(t)$$

Integrate: $\mu(t)\,y(t) = \int \mu(t)\,g(t)\,dt + C$

$$y(t) = \frac{\int \mu(t)\,g(t)\,dt + C}{\mu(t)}$$

Remarks

* This formula ONLY works for standard form.
  Before doing anything, get the standard form first!
* When integrating $p(t)$, no need to worry about the constant.
  It makes no difference to the final solution.
* The arbitrary constant in the gen. sol'n appears in a fraction as part of the numerator, not a pure constant.

Example 1: $y' + 2y = e^{3t}$, $y(0) = 3$

Already in std. form

Int. factor: $\mu(t) = e^{\int 2\,dt} = e^{2t} e^{c}$

Gen. sol'n: $y(t) = \frac{\int e^{2t}\cancel{e^{c}}e^{3t}\,dt}{e^{2t}\cancel{e^{c}}} = \frac{\int e^{5t}\,dt}{e^{2t}} = \frac{\frac{1}{5}e^{5t} + C}{e^{2t}}$

$$= \frac{1}{5}e^{3t} + Ce^{-2t}$$

Init. val: $3 = \frac{1}{5}e^{0} + Ce^{0} \Rightarrow C = \frac{14}{5}$

Soln to the IVP: $y(t) = \frac{1}{5}e^{3t} + \frac{14}{5}e^{-2t}$.

Supplementary problem: determine $t \to \infty$ behavior. (long term)

$t \to \infty$, $y(t) \to +\infty$

or $y(t) \sim \frac{1}{5}e^{3t}$.

Example 2: $ty' - y = t^3$. $y(1) = 0$

Std. form: $y' - \frac{1}{t}y = t^2$

Int. factor: $\mu(t) = e^{\int (-\frac{1}{t})\,dt} = e^{-\ln t} \doteq \cancel{(-t)}$ abuse of algebra

$-\ln t = \ln t^{-1}$ $= e^{\ln t^{-1}}$

$a \ln b = \ln b^{a}$

$e^{\ln a} = a$ $= t^{-1} = \frac{1}{t}$

Gen. sol'n: $y(t) = \dfrac{\int \frac{1}{t}\cdot t^2\,dt}{\frac{1}{t}} = \dfrac{\int t\,dt}{\frac{1}{t}} = \dfrac{\frac{1}{2}t^2 + C}{\frac{1}{t}}$

$$= t\left(\frac{1}{2}t^2 + C\right) = \frac{1}{2}t^3 + Ct$$

Init. val. $0 = \frac{1}{2} \times 1^3 + C \times 1 \Rightarrow C = -\frac{1}{2}$

Sol'n to the IVP: $y(t)=\frac{1}{2}t^3-\frac{1}{2}t$

Example 3: $(\sin t)\,y'+(\cos t)\,y=\sin^2 t$, $0<t<\pi$

Std. form: $y'+\frac{\cos t}{\sin t}y=\sin t$

Int. factor: $\mu(t)=e^{\int\frac{\cos t}{\sin t}dt}$

$$\int\frac{\cos t}{\sin t}dt \underset{\cos t\,dt=d\sin t}{=} \int\frac{d\sin t}{\sin t}=\ln|\sin t|$$

$\mu(t)=e^{\ln|\sin t|}=\sin t$ [I don't care about the const $\Rightarrow$ I don't care about the abs. val.]

Gen. sol'n: $y(t)=\dfrac{\int\sin t\cdot\sin t\,dt}{\sin t}=\dfrac{\int\sin^2 t\,dt}{\sin t}$

Recall: $\sin^2 t=\frac{1}{2}(1-\cos 2t)$

$$\int\sin^2 t\,dt=\int\left(\frac{1}{2}-\frac{1}{2}\cos 2t\right)dt=\frac{1}{2}t-\frac{1}{2}\cdot\left(\frac{1}{2}\sin 2t\right)$$

$$=\frac{1}{2}t-\frac{1}{4}\sin 2t$$

$$y(t)=\frac{\frac{1}{2}t-\frac{1}{4}\sin 2t+C}{\sin t}=\frac{t}{2\sin t}-\frac{\sin 2t}{4\sin t}+\frac{C}{\sin t}$$

$$\frac{\sin 2t}{4\sin t}=\frac{2\sin t\cos t}{4\sin t}=\frac{1}{2}\cos t.$$

The sol'n makes sense when $0<t<\pi$.

Example 4: $ty' + 2y = t(\ln 3t)^2$

Std. form: $y' + \frac{2}{t}y = (\ln 3t)^2$

Int. factor: $\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = t^2$

Gen. sol'n: $y(t) = \dfrac{\int t^2 \cdot (\ln 3t)^2 dt}{t^2}$

Integration by parts

$$\int \underbrace{f(t)}_{\text{DIFF}}\underbrace{g(t)}_{\text{INT}}\,dt = \underbrace{f(t)\,\overset{\int g(t)dt}{G(t)}}_{\text{Do ONLY INT}} - \int \underbrace{f'(t)\,G(t)}_{\text{Do BOTH DIFF \& INT.}}\,dt$$

$$\int \underbrace{t^2}_{\text{INT}}\underbrace{(\ln 3t)^2}_{\text{DIFF}}\,dt = \frac{1}{3}t^3(\ln 3t)^2 - \int \frac{1}{3}t^3 \cdot 2(\ln 3t)\cdot\frac{1}{3t}\cdot 3\,dt$$

$$= \frac{1}{3}t^3(\ln 3t)^2 - \frac{2}{3}\int \underbrace{t^2}_{\text{INT}}\underbrace{\ln 3t}_{\text{DIFF}}\,dt$$

$$= \frac{1}{3}t^3(\ln 3t)^2 - \frac{2}{3}\left[\frac{1}{3}t^3\ln 3t - \int \frac{1}{3}t^3\cdot\frac{1}{\cancel{3}t}\cdot\cancel{3}\,dt\right]$$

$$= \frac{1}{3}t^3(\ln 3t)^2 - \frac{2}{9}t^3\ln 3t + \frac{2}{9}\int t^2\,dt$$

$$= \frac{1}{3}t^3(\ln 3t)^2 - \frac{2}{9}t^3\ln 3t + \frac{2}{27}t^3 + C.$$

$$y(t) = \frac{1}{t^2}\left(\frac{1}{3}t^3(\ln 3t)^2 - \frac{2}{9}t^3\ln 3t + \frac{2}{27}t^3 + C\right)$$

$$y(t) = \frac{1}{3}t(\ln 3t)^2 - \frac{2}{9}t \ln 3t + \frac{2}{27}t + \frac{C}{t^2}$$

Example 5: $y' + y = \cos 2t$ $\int e^t \cos 2t$ Int. by part.

Example 6: $y' - \tan t \, y = \sec^2 t$ $\int \sec t = \ln|\sec t + \tan t| + C$

Check the review slides of basic formulas.

Solution for example 5.

Int. factor: $\mu(t) = e^{\int 1 dt} = e^t$

Gen. sol'n: $y(t) = \dfrac{\int e^t \cos 2t \, dt}{e^t}$

$$\int e^t \cos 2t \, dt = e^t \cos 2t - \int e^t(-2\sin 2t)\, dt$$

$$= e^t \cos 2t + 2\int e^t \sin 2t$$

$$= e^t \cos 2t + 2\left(e^t \sin 2t - \int e^t (2\cos 2t) dt\right)$$

$$\int e^t \cos 2t \, dt = e^t \cos 2t + 2e^t \sin 2t - 4\int e^t \cos 2t$$

$$5\int e^t \cos 2t\,dt = e^t\cos 2t + 2e^t \sin 2t$$

$$\int e^t \cos 2t\,dt = \frac{1}{5}\left(e^t\cos 2t + 2e^t\sin 2t\right)$$

$$y(t) = \frac{1}{e^t}\left(\frac{1}{5}e^t\cos 2t + \frac{2}{5}e^t \sin 2t + C\right)$$

$$= \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t + Ce^{-t}$$

Solution for example 6:

$$\mu(t) = e^{-\int \tan t\,dt} = e^{-\int \frac{\sin t}{\cos t}dt} = e^{\ln|\cos t|} = \cos t$$

$$y(t) = \frac{\int \cos t \cdot \sec^2 t\,dt}{\cos t} = \frac{1}{\cos t}\int \sec t\,dt$$

Check Review Slides $= \frac{1}{\cos t}\left(\ln|\sec t + \tan t| + C\right)$

$$= (\sec t)\ln|\sec t + \tan t| + C\sec t.$$